SOA/CAS MAY 2003 COURSE 1 EXAM SOLUTIONS

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1. We identify the following events:
G - watched gymnastics , B - watched baseball , S - watched soccer .
We wish to find $P[G' \cap B' \cap S']$. By DeMorgan's rules we have
$P[G' \cap B' \cap S'] = 1 - P[G \cup B \cup S]$.
We use the relationship

$$P[G \cup B \cup S] = P[G] + P[B] + P[S] - \big(P[G \cap B] + P[G \cap S] + P[B \cap S]\big) + P[G \cap B \cap S] .$$

We are given $P[G] = .28$, $P[B] = .29$, $P[S] = .19$,
$P[G \cap B] = .14$, $P[G \cap S] = .10$, $P[B \cap S] = .12$, $P[G \cap B \cap S] = .08$.
Then $P[G \cup B \cup S] = .48$ and $P[G' \cap B' \cap S'] = 1 - .48 = .52$. Answer: D

2. For each of the five possible answers at least on of the curves is a straight line. If the straight line was $f(x)$, then $f''(x) = 0$. This is true since a straight line would be of the form $f(x) = ax + b$, and $f'(x) = a$, $f''(x) = 0$. Therefore, the straight line must be $f''(x)$.
We can eliminate E since both curves are straight lines. We can eliminate C and D, since the second derivative of the convex curve is $f''(x) > 0$, but the two straight lines are negative for some values of x. We see that for $x < 0$ the curve in A is concave down, so that $f''(x) < 0$ for $x < 0$, and we see that for $x > 0$ the curve in A is convex up, so that $f''(x) > 0$ for $x > 0$. The straight line in A is < 0 for $x < 0$ and it is > 0 for $x > 0$.
This matches with the second derivative of the curve.
As an alternative approach, for each answer, the straight line goes through the origin, and must be of the form $f''(x) = mx$ (slope m). Therefore, $f(x) = \frac{m}{3}x^3 + cx + d$, since the second derivative of $cx + d$ is 0. This rules of answers C, D and E, since there are no cubic functions (C and D have quadratic looking functions and E has a linear function).
Since $f(x)$ also goes through the origin for each answer, it follows that $d = 0$, so that $f(x) = \frac{m}{3}x^3 + cx$ and $f(x) = mx$. If $m < 0$ (negative slope for the straight line $f''(x) = mx$) and then $\lim_{x \to \infty} f(x) = \frac{m}{3}x^3 + cx = -\infty$. This does not occur in A or B.
Therefore it must be true that $m > 0$ (positive slope). This occurs in answer A. Answer: A

3. $\lim_{x \to 0} \frac{cf(x)-dg(x)}{f(x)-g(x)} = \frac{c^2-d^2}{c-d} = \frac{(c-d)(c+d)}{c-d} = c + d$. Answer: E

4. The exponential time until failure random variable T has density function of the form $f(t) = \lambda e^{-\lambda t}$, and had distribution function $F(t) = P[T \leq t] = 1 - e^{-\lambda t}$ for $t > 0$.
The median of the distribution is the time point m that satisfies the relationship $F(m) = .5$; in other words, m is the time point for which there is a 50% probability of failure by time m. We are given that $m = 4$, and therefore $F(4) = 1 - e^{-4\lambda} = .5$, from which it follows that $e^{-4\lambda} = .5$. We are asked to find $P[T > 5] = 1 - F(5) = e^{-5\lambda}$.
Using the relationship $e^{-5\lambda} = (e^{-4\lambda})^{1.25}$, we get $P[T > 5] = e^{-5\lambda} = (.5)^{1.25} = .420$.
Notice that we could solve for λ from the equation $e^{-4\lambda} = .5$, but it is not necessary.
Answer: D

5. We identify the following events:
A - the policyholder insures exactly one car (so that A' is the event that the policyholder insures more than one car), and
S - the policyholder insures a sports car.
We are given $P[A'] = .7$ (from which it follows that $P[A] = .3$) , and $P[S] = .2$ (and $P[S'] = .8$). We are also given the conditional probability $P[S|A'] = .15$; "of those customers who insure more than one car", means that we are looking at a conditional event given A' .
We are asked to find $P[A \cap S']$.
We create the following probability table, with the numerals in parentheses indicating the order in which calculations are performed.

	A , .3	A' , .7
S , .2	**(2)** $P[S \cap A]$ $= P[S] - P[S \cap A']$ $= .2 - .105 = .095$	**(1)** $P[S \cap A'] = P[S \mid A'] \cdot P[A']$ $= (.15)(.7) = .105$
S' , .8	**(3)** $P[A \cap S']$ $= P[A] - P[A \cap S]$ $= .3 - .095 = .205$	

Answer: B

6. $Cov(X,Y) = E[XY] - E[X] \cdot E[Y]$.

We $E[X] = \int\int x \cdot f(x,y)\, dy\, dx$. Since the region of probability is defined with $x \le y \le 2x$, we apply double integration in the $dy\, dx$ order. It would be possible to reverse the order, but that would not make the solution any more efficient. $E[Y]$ and $E[XY]$ are found in a similar way.

$E[X] = \int_0^1 \int_x^{2x} x \cdot \frac{8}{3} \cdot xy\, dy\, dx = \frac{8}{3} \cdot \int_0^1 \int_x^{2x} x^2 y\, dy\, dx$

$= \frac{8}{3} \cdot \int_0^1 [\frac{x^2y^2}{2} \Big|_{y=x}^{y=2x}]\, dx = \frac{8}{3} \cdot \int_0^1 [\frac{3x^4}{2}]\, dx = \frac{8}{3} \cdot \frac{3}{2} \cdot \frac{1}{5} = \frac{4}{5}$.

$E[Y] = \int_0^1 \int_x^{2x} y \cdot \frac{8}{3} \cdot xy\, dy\, dx = \frac{8}{3} \cdot \int_0^1 \int_x^{2x} xy^2\, dy\, dx$

$= \frac{8}{3} \cdot \int_0^1 [\frac{xy^3}{3} \Big|_{y=x}^{y=2x}]\, dx = \frac{8}{3} \cdot \int_0^1 [\frac{7x^4}{3}]\, dx = \frac{8}{3} \cdot \frac{7}{3} \cdot \frac{1}{5} = \frac{56}{45}$.

$E[XY] = \int_0^1 \int_x^{2x} xy \cdot \frac{8}{3} \cdot xy\, dy\, dx = \frac{8}{3} \cdot \int_0^1 \int_x^{2x} x^2y^2\, dy\, dx$

$= \frac{8}{3} \cdot \int_0^1 [\frac{x^2y^3}{3} \Big|_{y=x}^{y=2x}]\, dx = \frac{8}{3} \cdot \int_0^1 [\frac{7x^5}{3}]\, dx = \frac{8}{3} \cdot \frac{7}{3} \cdot \frac{1}{6} = \frac{28}{27}$.

Then $Cov(X,Y) = \frac{28}{27} - (\frac{4}{5})(\frac{56}{45}) = .041$. Answer: A

7. To find $\int_0^2 f(2x)\, dx$ we apply the change of variable $u = 2x$, so that $du = 2\, dx$, or equivalently, $dx = \frac{1}{2}\, du$. Then $\int_0^2 f(2x)\, dx = \int_0^4 f(u)\, \frac{1}{2}\, du = \frac{1}{2} \cdot \int_0^4 f(u)\, du$ (since $u = 2x$, we must adjust the limits of integration in the transformed integral).

Variables of integration are "dummy variables", and therefore

$\int_0^4 f(u)\, du = \int_0^4 f(t)\, dt = \int_0^4 f(x)\, dx = \int_0^2 f(x)\, dx + \int_2^4 f(x)\, dx = 3 + 5$

(it doesn't matter what letter is used for the integration variable).

Finally, $\int_0^2 f(2x)\, dx = \frac{1}{2} \cdot \int_0^4 f(u)\, du = \frac{1}{2} \cdot (3+5) = 4$. Answer: C

8. We identify the following events:

A - the driver has an accident ,

T (teen) - age of driver is 16-20 , Y (young) - age of driver is 21-30 ,

M (middle age) - age of driver is 31-65 , S (senior) - age of driver is 66-99 .

The final column in the table lists the probabilities of T, Y, M and S , and the middle column gives the conditional probability of A given driver age. The table can be interpreted as

Age	Probability of Accident	Portion of Insured Drivers
16-20	$P[A\|T] = .06$	$P[T] = .08$
21-30	$P[A\|Y] = .03$	$P[Y] = .15$
31-65	$P[A\|M] = .02$	$P[M] = .49$
66-99	$P[A\|S] = .04$	$P[S] = .28$

We are asked to find $P[T|A]$.

We construct the following probability table, with numerals in parentheses indicating the order of the calculations.

	$T, .08$	$Y, .15$	$M, .49$	$S, .28$
A	**(1)** $P[A\cap T]$ $= P[A\|T]\cdot P[T]$ $= (.06)(.08)$ $= .0048$	**(2)** $P[A\cap Y]$ $= P[A\|Y]\cdot P[Y]$ $= (.03)(.15)$ $= .0045$	**(3)** $P[A\cap M]$ $= P[A\|M]\cdot P[M]$ $= (.02)(.49)$ $= .0098$	**(4)** $P[A\cap S]$ $= P[A\|M]\cdot P[M]$ $= (.04)(.28)$ $= .0112$

(5) $P[A] = P[A\cap T] + P[A\cap Y] + P[A\cap M] + P[A\cap S] = .0303$

(6) $P[T|A] = \frac{P[A\cap T]}{P[A]} = \frac{.0048}{.0303} = .158$. Answer: B

9. To say that the payment is 20% less than the previous year's payment is the same as saying that the payment is 80% of the previous year's payments. The successive yearly payments will be 60 , $60(.8)$, $60(.8)^2$, $60(.8)^3$, . . . This forms a geometric series.

The total claims paid in all years after the company stops selling malpractice insurance is

$60 + 60(.8) + 60(.8)^2 + 60(.8)^3 + \cdots = 60[1 + (.8) + (.8)^2 + (.8)^3 + \cdots] = 60\cdot\frac{1}{1-.8} = 300$.

Answer: D

10. The range $x^2 \le y \le x$ is only valid for $0 \le x \le 1$. This is true since $x^2 > x$ for $x > 1$ and $x^2 > 0 > x$ for $x < 0$. Therefore, the range for y is $0 \le x^2 \le y \le x \le 1$, so that $0 \le y \le 1$. Also, the inequality $x^2 \le y$ is equivalent to $x \le \sqrt{y}$, so that $x^2 \le y \le x$ is equivalent to $y \le x \le \sqrt{y}$. The marginal density function of Y is found by integrating the joint density over the range for the other variable x;
$g(y) = \int_y^{\sqrt{y}} 15y\,dx = 15yx\Big|_{x=y}^{x=\sqrt{y}} = 15y(\sqrt{y} - y) = 15(y^{3/2} - y^2)$ for $0 \le y \le 1$.
Note that it is true that for any particular x we have $x^2 \le y \le x$. However, since x can be any number from 0 to 1, y can also be any number from 0 to 1. Answer: E

11. We apply the chain rule. In this case, the derivative of an exponential function is equal to that exponential function multiplied by the derivative of the exponent.
$S'(t) = 5000e^{0.1(e^{0.25t})} \cdot (0.1)(0.25e^{0.25t})$.
Then $S'(8) = 5000e^{0.1(e^2)} \cdot (0.1)(0.25e^2) = 1,934$. Answer: B

12. $E[X] = \int_{-2}^{4} x \cdot f(x)\,dx = \int_{-2}^{4} x \cdot \frac{|x|}{10}\,dx$.
For $x < 0$, $|x| = -x$ and for $x > 0$, $|x| = x$.
Then, $E[X] = \int_{-2}^{0} x \cdot (-\frac{x}{10})\,dx + \int_0^4 x \cdot (\frac{x}{10})\,dx = (.1)(-\int_{-2}^{0} x^2\,dx + \int_0^4 x^2\,dx)$
$= (.1)\big[-\frac{0^3-(-2)^3}{3} + \frac{4^3-0^3}{3}\big] = \frac{28}{15}$. Answer: D

13. The standard approximation to the sum (total) of a collection of independent random variables is the normal approximation. The total contribution is $T = C_1 + C_2 + \cdots + C_{2025}$, the sum of the 2025 contributions. C_i is the amount of the i-th contribution, the C_i's are mutually independent, and each has mean $E[C_i] = 3125$ and variance $Var[C_i] = (250)^2$.

The mean and variance of T are $E[T] = \sum_{i=1}^{2025} E[C_i] = (2025)(3125) = 6,328,125$ and

$Var[T] = \sum_{i=1}^{2025} Var[C_i] = (2025)(250^2) = 126,562,500$.

We will denote the 90th percentile of T by p. We find the approximate 90th percentile of T by applying the normal approximation to T. We wish to find p so that $P[T \leq p] = .9$.

We standardize the probability: $P[T \leq p] = P[\frac{T-6,328,125}{\sqrt{126,562,500}} \leq \frac{p-6,328,125}{\sqrt{126,562,500}}] = .90$.

$\frac{T-6,328,125}{\sqrt{126,562,500}}$ is approximately standard normal (mean 0, variance 1), so that

$\frac{p-6,328,125}{\sqrt{126,562,500}}$ is the 90-th percentile of the standard normal distribution. From the table for the standard normal distribution, we see that $\Phi(1.282) = .90$. Therefore we have

$\frac{p-6,328,125}{\sqrt{126,562,500}} = 1.282$, from which we get $p = 6,342,547.5$. Answer: C

14. Using the product rule for differentiation, we have

$g'(x) = \frac{d}{dx}[e^{-x} f(x)] = -e^{-x} f(x) + e^{-x} f'(x) = e^{-x}[f'(x) - f(x)]$, so that

$g'(3) = e^{-3}[f'(3) - f(3)]$.

We use the definition of derivative to find

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{3x^2h+3xh^2+h^3+2h}{h} = \lim_{h \to 0}[3x^2 + 3xh + h^2 + 2] = 3x^2 + 2$.

Then, $f'(3) = 3(3^2) + 2 = 29$.

Also, if we let $h = 3$ and $x = 0$, then $f(x+h) - f(x) = f(0+3) - f(0) = f(3) - 1$.

However, $f(0+3) - f(0) = 3(0^2)(3) + 3(0)(3) + 3^3 + 2(3) = 33$.

Therefore, $f(3) - 1 = 33$, so that $f(3) = 34$.

Finally, $g'(3) = e^{-3}[29 - 34] = -5e^{-3}$. Answer: C

15. The new amount paid to the surgeon is $X' = X + 100$, and the new amount of hospital charges is $Y' = 1.1Y$. We wish to find

$Var[X' + Y'] = Var[X'] + Var[Y'] + 2Cov(X', Y')$.

$Var[X'] = Var[X + 100] = Var[X] = 5,000$, and

$Var[Y'] = Var[1.1Y] = (1.1^2)Var[Y] = (1.21)(10,000) = 12,100$.

$Cov(X', Y') = Cov(X + 100, 1.1Y) = 1.1Cov(X, Y)$.

We have used the covariance rule $Cov(aU + b, cW + d) = acCov(U, W)$.

We still must know $Cov(X, Y)$ to complete the problem.

We are given $Var[X + Y] = 17,000$, and we use the relationship

$17,000 = Var[X + Y] = Var[X] + Var[Y] + 2Cov(X, Y)$

$= 5,000 + 10,000 + 2Cov(X, Y) \rightarrow Cov(X, Y) = 1,000$.

Then $Cov(X', Y') = 1.1Cov(X, Y) = 1,100$.

Finally, $Var[X' + Y'] = Var[X'] + Var[Y'] + 2Cov(X', Y')$

$= 5,000 + 12,100 + 2(1,100) = 19,300$. Answer: C

16. The device fails as soon as either component fails. The probability of failure within the first hour is $P[(X \leq 1) \cup (Y \leq 1)]$. There are a couple of ways in which this can be found.

We can use the probability rule

$P[(X \leq 1) \cup (Y \leq 1)] = P[X \leq 1] + P[Y \leq 1] - P[(X \leq 1) \cap (Y \leq 1)]$,

but this will require three separate double integrals (although the first two are equal because of the symmetry of the distribution).

Alternatively, we can use DeMorgan's rule, $P[A \cup B] = 1 - P[A' \cap B']$, so that

$P[(X \leq 1) \cup (Y \leq 1)] = 1 - P[(X > 1) \cap (Y > 1)]$.

Since both X and Y are between 0 and 3, we get

$P[(X > 1) \cap (Y > 1)] = \int_1^3 \int_1^3 (\frac{x+y}{27})\, dy\, dx = \frac{1}{27} \cdot \int_1^3 [(xy + \frac{1}{2}y^2)\Big|_{y=1}^{y=3}]\, dx$

$= \frac{1}{27} \cdot \int_1^3 (2x + 4)\, dx = \frac{1}{27} \cdot (x^2 + 4x)\Big|_{x=1}^{x=3} = \frac{16}{27}$.

Then, $P[(X \leq 1) \cup (Y \leq 1)] = 1 - \frac{16}{27} = \frac{11}{27} = .407$. Answer: B

17. $e^{1/n}+e^{2/n}+\cdots+e^{n/n}=e^{1/n}\cdot[1+e^{1/n}+e^{2/n}+\cdots+e^{(n-1)/n}]$

$= e^{1/n}\cdot\frac{e^{n/n}-1}{e^{1/n}-1}=\frac{e^{n/n}-1}{1-e^{-1/n}}=\frac{e-1}{1-e^{-1/n}}$.

We have used the formula for the sum of a geometric series:

$1+r+r^2+\cdots+r^k=\frac{r^{k+1}-1}{r-1}$. In this case, for the expression inside brackets,

$r=e^{1/n}$ and $k=n-1$.

We wish to find

$\lim_{n\to\infty}\frac{1}{n}\cdot[e^{1/n}+e^{2/n}+\cdots+e^{n/n}]=\lim_{n\to\infty}\frac{1}{n}\cdot[\frac{e-1}{1-e^{-1/n}}]$.

This will be equal to $\frac{e-1}{\lim_{n\to\infty}n\cdot(1-e^{-1/n})}$.

To find $\lim_{n\to\infty}n\cdot(1-e^{-1/n})$, we let $h=\frac{1}{n}$, so that the limit is equal to $\lim_{h\to 0}\frac{1-e^{-h}}{h}$,

and using l'Hospital's rule we get $\lim_{h\to 0}\frac{1-e^{-h}}{h}=\lim_{h\to 0}\frac{e^{-h}}{1}=1$.

Therefore, $\lim_{n\to\infty}\frac{1}{n}\cdot[e^{1/n}+e^{2/n}+\cdots+e^{n/n}]=e-1$. Answer: C

18. We define the following events

R - renew at least one policy next year

A - has an auto policy , H - has a homeowner policy

A policyholder with an auto policy only can be described by the event $A\cap H'$, and
a policyholder with a homeowner policy only can be described by the event $A'\cap H$.

We are given $P[R|A\cap H']=.4$, $P[R|A'\cap H]=.6$ and $P[R|A\cap H]=.8$.

We are also given $P[A]=.65$, $P[H]=.5$ and $P[A\cap H]=.15$.

We are asked to find $P[R]$.

We use the rule

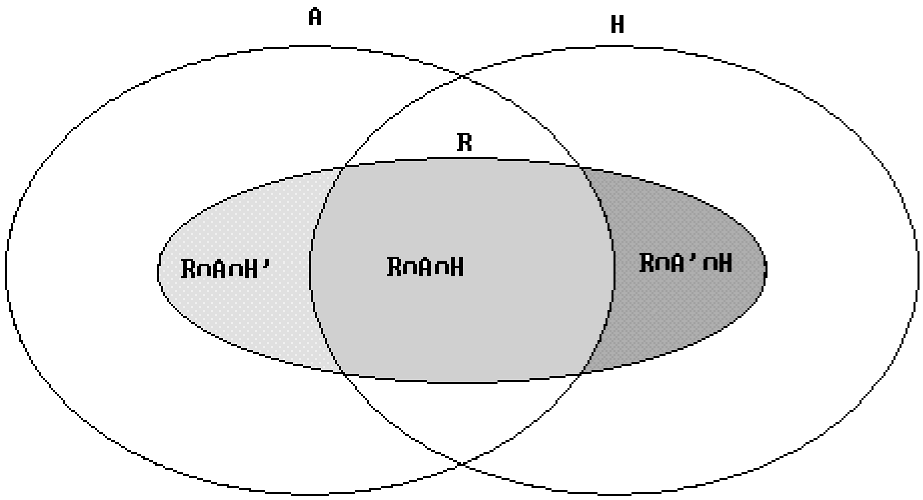
$P[R]=P[R\cap A\cap H]+P[R\cap A'\cap H]+P[R\cap A\cap H']+P[R\cap A'\cap H']$.

Since renewal can only occur if there is at least one policy, it follows that $P[R\cap A'\cap H']=0$; in other words, of there is no auto policy (event A') <u>and</u> there is no homeowner policy (event H'), then there can be no renewal. An alternative way of saying the same thing is that R is a subset (subevent) of $A\cup H$.

(Note also that $P[A\cup H]=P[A]+P[H]-P[A\cap H]=.65+.5-.15=1$, so this also show that R must be a subevent of $A\cup H$, and it also shows that $P[A'\cap H']=1-P[A\cup H]=1-1=0$ so that $A'\cap H'=\phi$).

This can be illustrated in the following diagram.

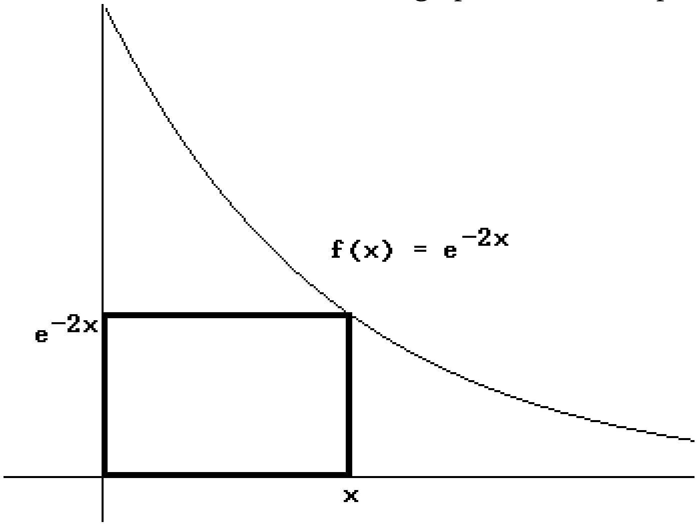
18. (continued)



We find $P[R \cap A \cap H]$, $P[R \cap A' \cap H]$ and $P[R \cap A \cap H']$ by using the rule
$P[C \cap D] = P[C|D] \cdot P[D]$:
$P[R \cap A \cap H] = P[R|A \cap H] \cdot P[A \cap H] = (.8)(.15) = .12$,
$P[R \cap A' \cap H] = P[R|A' \cap H] \cdot P[A' \cap H] = (.6)P[A' \cap H]$,
$P[R \cap A \cap H'] = P[R|A \cap H'] \cdot P[A \cap H'] = (.4)P[A \cap H']$.

In order to complete the calculations we must find $P[A' \cap H]$ and $P[A \cap H']$.
From the diagram above, or using the probability rule, we have
$P[A] = P[A \cap H] + P[A \cap H'] \rightarrow .65 = .15 + P[A \cap H'] \rightarrow P[A \cap H'] = .5$, and
$P[H] = P[A \cap H] + P[A' \cap H] \rightarrow .5 = .15 + P[A' \cap H] \rightarrow P[A' \cap H] = .35$.
Then $P[R \cap A' \cap H] = (.6)(.35) = .21$ and $P[R \cap A \cap H'] = (.4)(.5) = .2$.
Finally, $P[R] = .12 + .21 + .2 = .53$. 53% of policyholders will renew. Answer: D

19. The darkened border in the graph below is the perimeter of the rectangle.



The perimeter is $P(x) = 2(x + e^{-2x})$, and we are told that the interval for x is $x > 0$.
As $x \rightarrow 0$, the perimeter approaches $2(0 + e^0) = 2$, and as $x \rightarrow \infty$, the perimeter approaches ∞.
The critical point(s) for $P(x)$ are the solution(s) of $P'(x) = 2(1 - 2e^{-2x}) = 0$. The solution occurs at $x = -\frac{1}{2}\,ln(\frac{1}{2}) = .3466$. If $x = -\frac{1}{2}\,ln(\frac{1}{2}) = ln(\sqrt{2})$, the perimeter is $2(.3466 + e^{-2\,ln\sqrt{2}}) = 2(.3466 + .5) = 1.69$. The function $P(x)$ has an absolute minimum at $x = .3466$, but there is no absolute maximum since $\lim_{x\to\infty} P(x) = \infty$. Answer: A

20. Let X and Y denote the two loss amounts (<u>not</u> payment amounts).
We consider the following combinations of X and Y that result in the total benefit <u>payment</u> not exceeding 5.
Case 1: $0 < X \leq 1$ (so loss X results in no payment) and $0 < Y \leq 7$ (so that loss Y results in a maximum payment of 5 after applying the deductible of 2).
Case 2: $1 < X \leq 6$ (so loss X results in a maximum payment of 5 after the deductible of 1 is applied) and $0 < Y \leq 2$ (so loss Y results in no payment).
Case 3: $1 < X \leq 6$ and $2 \leq Y \leq 7$ and $(X-1)+(Y-2) \leq 5$ ($X - 1$ is paid for loss X and $Y - 2$ is paid for loss Y). The last condition is equivalent to $X + Y \leq 8$.
The probability that the total benefit paid does not exceed 5 is the sum of the probabilities for Cases 1, 2 and 3.

$P[\text{Case 1}] = P[(0 < X \leq 1) \cap (0 < Y \leq 7)]$

$= P[0 < X \leq 1] \cdot P[0 < Y \leq 7] = (\frac{1}{10})(\frac{7}{10}) = .07$

(we have used the independence of X and Y to find the probability of the intersection)

20. (continued)

P[Case 2] $= P[(1 < X \le 6) \cap (0 < Y \le 2)]$

$= P[1 < X \le 6] \cdot P[0 < Y \le 2] = (\frac{5}{10})(\frac{2}{10}) = .10$.

P[Case 3] $= \int_1^6 \int_2^{8-x} f(x,y)\,dy\,dx = \int_1^6 \int_2^{8-x} f_X(x) \cdot f_Y(y)\,dy\,dx$

$= \int_1^6 \int_2^{8-x} (\frac{1}{10})(\frac{1}{10})\,dy\,dx = \int_1^6 \int_2^{8-x} (.01)\,dy\,dx$

$= (.01)\int_1^6 [8 - x - 2]\,dx = (.01)(6x - \frac{x^2}{2}\Big|_{x=1}^{x=6}) = .125$

(note that $f(x,y) = f_X(x) \cdot f_Y(y)$ because X and Y are independent).

The total probability is $.07 + .10 + .125 = .295$.

Once we have identified Cases 1, 2 and 3, this problem could be approached from a graphical point of view. Since X and Y are independent and uniform, the joint distribution of X and Y is uniform on the square $0 < x < 10$, $0 < y < 10$, with joint density $(.1)(.1) = .01$.

Since the joint distribution is uniform, the probability of any event involving X and Y is equal to the constant density (.01 in this case) multiplied by the area of the region representing the event.

The three regions for Cases 1, 2 and 3 are indicated in the graph below.

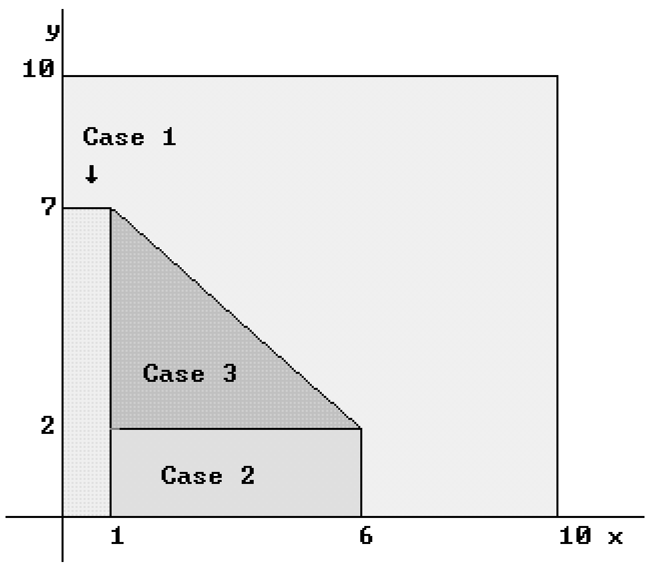
The 10×10 square is the full region for the joint distribution.

The rectangular area for Case 1 is $1 \times 7 = 7$ for a probability of $7 \times .01 = .07$.

The rectangular area for Case 2 is $5 \times 2 = 10$ for a probability of $10 \times .01 = .10$.

The triangular are for Case 3 is $\frac{1}{2} \times 5 \times 5 = 12.5$ for a probability of $12.5 \times .01 = .125$.

The total probability for Cases 1, 2 and 3 combined is again $.295$.



Answer: C

21. Labor and capital are both functions of time, with $\frac{dL}{dt} = 2.5$ and $\frac{dC}{dt} = .5$.
We apply the product and chain rule to get
$\frac{dP}{dt} = \frac{d}{dt}(3.5L^{6/5}C^{1/2}) = (3.5)[\frac{6}{5}L^{1/5} \cdot \frac{dL}{dt} \cdot C^{1/2} + L^{6/5} \cdot \frac{1}{2} \cdot C^{-1/2} \cdot \frac{dC}{dt}]$
$= (3.5)[\frac{6}{5} \cdot (12)^{1/5}(2.5)(4)^{1/2} + (12)^{6/5}(\frac{1}{2})(4)^{-1/2}(.5)] = 43.1$. Answer: D

22. The 30-th percentile of X, say c, is the point for which $F(c) = P[X \leq c] = .3$.
Therefore, $.3 = \int_{200}^{c} \frac{2.5(200)^{2.5}}{x^{3.5}}\,dx = -\frac{(200)^{2.5}}{x^{2.5}}\Big|_{x=200}^{x=c} = 1 - (\frac{200}{c})^{2.5} = .3$.
Solving for c results in $c = 230.7$.
The 70-th percentile of X, say d, is the point for which $P[X \leq d] = .7$.
Therefore, $.7 = \int_{200}^{d} \frac{2.5(200)^{2.5}}{x^{3.5}}\,dx = -\frac{(200)^{2.5}}{x^{2.5}}\Big|_{x=200}^{x=d} = 1 - (\frac{200}{d})^{2.5} = .7$.
Solving for d results in $d = 323.7$. Then $d - c = 93$. Answer: B

23. The density function for Y is $f_Y(y)$. If we can find $F_Y(y)$, the cumulative distribution function for Y then $f_Y(y) = F_Y'(y)$. We can find $F_Y(y)$ from the relationship between Y and T and from $F_T(t)$ (the cdf of T).
$F_Y(y) = P[Y \leq y] = P[T^2 \leq y] = P[0 < T \leq \sqrt{y}\,]$
(the description of $F_T(t)$ indicates that T is defined for only positive numbers).
Therefore, $F_Y(y) = F_T(\sqrt{y}) = 1 - (\frac{2}{\sqrt{y}})^2 = 1 - \frac{4}{y}$.
The density function for Y is $f_Y(y) = F_Y'(y) = \frac{4}{y^2}$. Answer: A

24. $E[X] = \int_2^{10}\int_0^1 x \cdot \frac{1}{64}(10 - xy^2)\,dy\,dx$.
The "inside" integral is
$\int_0^1 x \cdot \frac{1}{64}(10 - xy^2)\,dy = \frac{1}{64} \cdot \int_0^1 (10x - x^2y^2)\,dy = \frac{1}{64} \cdot [10x - \frac{x^2}{3}]$.
The complete integral is
$\int_2^{10} \frac{1}{64} \cdot [10x - \frac{x^2}{3}]\,dx = \frac{1}{64} \cdot [(5x^2 - \frac{x^3}{9})\Big|_{x=2}^{x=10}] = 5.8$.
Note that we could have found $f_X(x)$, the marginal density function of X first and the have found $E[X]$. This would be done as follows:
$f_X(x) = \int_0^1 \frac{1}{64}(10 - xy^2)\,dy = \frac{1}{64}(10 - \frac{x}{3})$, and then
$E[X] = \int_2^{10} x \cdot f_X(x)\,dx = \int_2^{10} x \cdot \frac{1}{64}(10 - \frac{x}{2})\,dx = \frac{1}{64} \cdot \int_2^{10}(10x - \frac{x^2}{3})\,dx$
$= \frac{1}{64} \cdot (5x^2 - \frac{x^3}{9})\Big|_{x=2}^{x=10}] = 5.8$ (as in the first approach). Answer: C

25. The insurance payment is $Y = \begin{cases} 0 & X \le c \\ X - C & C < X < 1 \end{cases}$.

The insurance payment is less than .5 if the $X - C < .5$, or equivalently, if $X < C + .5$.

It must be true that $C \le .5$, because if $C > .5$ then $C + .5 > 1$ and then $P[X < C + .5] = 1$ since $P[X < 1] = 1$.

$P[X < C + .5] = \int_0^{c+.5} 2x\, dx = (c + .5)^2$. In order for this to be equal to .64 we must have $(c + .5)^2 = .64$ → $c + .5 = .8$ (we ignore the negative square root since $X > 0$) → $c = .3$.

Answer: B

26. $g(x) = \frac{x+4}{x^2+2x-8} = \frac{x+4}{(x+4)(x-2)}$.

It is tempting to cancel the $x + 4$ factor and write $g(x) = \frac{1}{x-2}$.

This is not correct. $g(x)$ is equal to $\frac{1}{x-2}$, except at the point $x = -4$.

At that point $g(-4) = \frac{0}{0}$, which is not defined.

The graph of $g(x)$ would be identical to the graph of $\frac{1}{x-2}$, except that there would be an empty space at $x = -4$. $g(x)$ is discontinuous at $x = -4$ since $g(-4)$ is not defined , but it is possible to define $g(-4) = \frac{1}{-4-2} = -\frac{1}{6}$, and then $g(x)$ would be continuous at $x = -4$. Since $\lim_{x\to 2} g(x)$ is infinite (actually, $\lim_{x\to 2^+} g(x) = +\infty$ and $\lim_{x\to 2^-} g(x) = -\infty$), there is no way to define $g(2)$ that would make $g(x)$ continuous at $x = 2$. Answer: A

27. The differential equation can be written in the form $\frac{1}{A}\,\frac{dA}{dt} = i$.

This can be written as $\frac{d}{dt}\, ln\, A(t) = i$.

For this differential equation, $A(t) = Ke^{it}$.

The initial investment of 5,000 is made at time 0 and we want the investment to grow to 20,000 in 24 years. Therefore $A(0) = 5000 = Ke^0$ and $A(24) = Ke^{24i} = 20,000$.

Then $\frac{20,000}{5,000} = \frac{Ke^{24i}}{K} = e^{24i}$ → $i = \frac{1}{24} \cdot ln(4) = \frac{1}{24} \cdot ln(2^2) = \frac{2}{24} \cdot ln(2) = \frac{1}{12} \cdot ln(2)$.

Answer: D

28. If we find the conditional density function $f_{Y|X}(y|X=\frac{1}{3})$, then
$P[Y<X|X=\frac{1}{3}]=P[Y<\frac{1}{3}|X=\frac{1}{3}]=\int_0^{1/3} f_{Y|X}(y|X=\frac{1}{3})\,dy$.
The conditional density is $f_{Y|X}(y|X=\frac{1}{3})=\frac{f(\frac{1}{3},y)}{f_X(\frac{1}{3})}$.
The joint density is $f(\frac{1}{3},y)=24(\frac{1}{3})y=8y$, $0<y<1-\frac{1}{3}$,
and the marginal density of X at $X=\frac{1}{3}$ is $f_X(\frac{1}{3})=\int_0^{2/3} 24(\frac{1}{3})y\,dy=\frac{16}{9}$.
The conditional density is $f_{Y|X}(y|X=\frac{1}{3})=\frac{8y}{\frac{16}{9}}=\frac{9y}{2}$.
The conditional probability is $P[Y<X|X=\frac{1}{3}]=\int_0^{1/3}\frac{9y}{2}\,dy=\frac{1}{4}$. Answer: C

29. $F(v)=P[V\le v]=P[10{,}000e^R\le v]=P[R\le ln(\frac{v}{10{,}000})]$.
If X has a uniform distribution on the interval (a,b) then if $a<x<b$,
$P[X\le x]=\frac{x-a}{b-a}$. Since R is uniform on $(.04,.08)$ it follows that
$P[R\le ln(\frac{v}{10{,}000})]=\frac{ln(\frac{v}{10{,}000})-.04}{.08-.04}=25[ln(\frac{v}{10{,}000})-.04]$ if $.04<ln(\frac{v}{10{,}000})<.08$.
Answer: E

30. The sequence $\frac{k(k+1)}{2}$ for $k=0,1,2,\ldots$ has the following pattern

k :	0	1	2	3	4	5	6	7	8	9	$\cdots$
$\frac{k(k+1)}{2}$:	0	1	3	6	10	15	21	28	36	45	$\cdots$
	even	odd	odd	even	even	odd	odd	even	even	odd	$\cdots$
$(-1)^{k(k+1)/2}$	1	-1	-1	1	1	-1	-1	1	1	-1	$\cdots$

The summation $\sum_{k=0}^{\infty}(-1)^{k(k+1)/2}x^k$ can be broken into 4 summations.
The first summation is for $k=0,4,8,12,\ldots$, so that $(-1)^{k(k+1)/2}=1$.
The second summation is for $k=1,5,9,13,\ldots$, $(-1)^{k(k+1)/2}=-1$.
The third summation is for $k=2,6,10,14,\ldots$, $(-1)^{k(k+1)/2}=-1$.
The fourth summation is for $k=3,7,11,15,\ldots$, $(-1)^{k(k+1)/2}=1$.
The first summation is $x^0+x^4+x^8+x^{12}+\cdots=\frac{1}{1-x^4}$.
The second summation is $-[x^1+x^5+x^9+x^{13}+\cdots=\frac{-x}{1-x^4}$.
The third summation is $-[x^2+x^6+x^{10}+x^{14}+\cdots=\frac{-x^2}{1-x^4}$.
The fourth summation is $x^3+x^7+x^{11}+x^{15}+\cdots=\frac{x^3}{1-x^4}$.
The total summation is the combination of all four summations. This is

$$\sum_{k=0}^{\infty}(-1)^{k(k+1)/2}x^k=\frac{1}{1-x^4}-\frac{x}{1-x^4}-\frac{x^2}{1-x^4}+\frac{x^3}{1-x^4}$$

$$=\frac{1-x-x^2+x^3}{1-x^4}=\frac{(1-x)(1-x^2)}{(1+x^2)(1-x^2)}=\frac{1-x}{1+x^2}.$$ Answer: E

31. We identify the following events
N - non-smoker , L - light smoker , H - heavy smoker ,
D - dies during the 5-year study .
We are given $P[N] = .50$, $P[L] = .30$, $P[H] = .20$.
We are also told that $P[D|L] = 2P[D|N] = \frac{1}{2}P[D|H]$
(the probability that a light smoker dies during the 5-year study period is $P[D|L]$;
it is the conditional probability of dying during the period given that the individual is a light smoker). We wish to find the conditional probability $P[H|D]$.
We will find this probability from the basic definition of conditional probability,
$P[H|D] = \frac{P[H \cap D]}{P[D]}$. These probabilities can be found from the following probability table.
The numerals indicate the order in which the calculations are made.
We are not given specific values for $P[D|L]$, $P[D|N]$, or $P[D|H]$, so will let $P[D|N] = k$, and then $P[D|L] = 2k$ and $P[D|H] = 4k$.

	N , .5	L , .3	H , .2
D	**(1)** $P[D \cap N]$ $= P[D|N] \cdot P[N]$ $= (k)(.5) = .5k$	**(2)** $P[D \cap L]$ $= P[D|L] \cdot P[L]$ $= (2k)(.3) = .6k$	**(3)** $P[D \cap H]$ $= P[D|H] \cdot P[H]$ $= (4k)(.2) = .8k$

(4) $P[D] = P[D \cap N] + P[D \cap L] + P[D \cap H] = .5k + .6k + .8k = 1.9k$.
(5) $P[H|D] = \frac{P[H \cap D]}{P[D]} = \frac{.8k}{1.9k} = .42$. Answer: D

32. From the definition of $F(x)$ we see that $F(1) = \frac{1}{2}$. This indicates that X has a point of probability at $X = 1$ with $P[X = 1] = \frac{1}{2}$. For $1 < x < 2$, the density function for X is $f(x) = F'(x) = x - 1$. We formulate the variance of X as $Var[X] = E[X^2] - (E[X])^2$.
$E[X] = (1) \cdot P[X = 1] + \int_1^2 x \cdot f(x)\, dx = (1)(\frac{1}{2}) + \int_1^2 x(x-1)\, dx = \frac{1}{2} + \frac{7}{3} - \frac{3}{2} = \frac{4}{3}$.
$E[X^2] = (1^2) \cdot P[X = 1] + \int_1^2 x^2 \cdot f(x)\, dx = (1)(\frac{1}{2}) + \int_1^2 x^2(x-1)\, dx = \frac{1}{2} + \frac{15}{4} - \frac{7}{3} = \frac{23}{12}$.
$Var[X] = \frac{23}{12} - (\frac{4}{3})^2 = \frac{5}{36}$. Answer: C

33. We denote $f^2(x) = f(f(x)) = a_2$, $f^3(x) = f(f^2(x)) = a_3$, ... ,
$f^n(x) = f(f^{n-1}(x)) = f(a_{n-1}) = a_n$.
We wish to find $\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} a_n = L$.
If the limit L exists, then $L = \lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} f(a_{n-1}) = f(L) = \frac{2L}{L+1}$.
Therefore, $L = \frac{2L}{L+1}$, and the solutions to this equation are $L = 0$ and $L = 1$.
If $0 < c < 1$ then $f(c) = \frac{2c}{c+1} > \frac{2c}{1+1} = c$;
therefore if $1 > a_{n-1} > 0$ then $a_n = f(a_{n-1}) > a_{n-1}$.
This indicates that the limit cannot be 0, since anytime $0 < a_{n-1} < 1$, the next term in the series is $a_n > a_{n-1}$. Therefore, the limit must be 1. Answer: B

34. $f(x) = c(10+x)^{-2}$, $0 < x < 40$.
The total probability must be 1, so that $\int_0^{40} c(10+x)^{-2}\,dx = c[\frac{1}{10} - \frac{1}{50}] = 1$.
Therefore, $c = 12.5$ and $f(x) = 12.5(10+x)^{-2}$.
Then, $P[X < 6] = \int_0^6 f(x)\,dx = \int_0^6 12.5(10+x)^{-2}\,dx = -12.5(10+x)^{-1}\Big|_{x=0}^{x=6} = .46875$.
Answer: C

35. We can use a version of the second derivative test for a function of two variables.
Given the function $f(x,y)$, we define Δ :

$$\Delta = \left[\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2\right]$$

If $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} = 0$, then

1. if $\Delta > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, then f has a relative minimum at (x_0, y_0)
2. if $\Delta > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$, then f has a relative maximum at (x_0, y_0)
3. if $\Delta < 0$, then f has neither a relative minimum or maximum at (x_0, y_0)

For the given function, $\frac{\partial^2 f}{\partial x^2} = 24x - 4y + 40$, $\frac{\partial^2 f}{\partial y^2} = 2$, $\frac{\partial^2 f}{\partial x \partial y} = -4x$.
Then, $\Delta = (24x - 4y + 40)(2) - (-4x)^2 = -16x^2 + 48x - 8y + 80$.
At the indicated critical points, the values of $\frac{\partial^2 f}{\partial x^2}$ and Δ are

(x,y)	$\frac{\partial^2 f}{\partial x^2}$	Δ	Local Max. or Min.
$(-2,4)$	$-24 < 0$	$-112 < 0$	No local max. or min.
$(0,0)$	$40 > 0$	$80 > 0$	Local minimum
$(5,25)$	$60 > 0$	$-280 < 0$	No local max. or min.

Answer: E

36. The probability function is

x :	1	2	3	4	5
$P[X=x]$:	$\frac{5}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
Amount paid :	100	200	300	325	350

Expected amount paid
$=(100)(\frac{5}{15})+(200)(\frac{4}{15})+(300)(\frac{3}{15})+(325)(\frac{2}{15})+(350)(\frac{1}{15})=213.3$.
Answer: D

37. We define the following events.
E - the claim includes emergency room charges ,
O - the claim includes operating room charges.
We are given $P[E\cup O]=.85$, $P[E']=.25$ and E and O are independent.
We are asked to find $P[O]$.
We use the probability rule $P[E\cup O]=P[E]+P[O]-P[E\cap O]$.
Since E and O are independent, we have $P[E\cap O]=P[E]\cdot P[O]=(.75)P[O]$
(since $P[E]=1-P[E']=1-.25=.75$).
Therefore, $.85=P[E\cup O]=.75+P[O]-.75P[O]$.
Solving for $P[O]$ results in $P[O]=.40$. Answer: D

38. Since the inventory will be replenished when it drops to 1 and since inventory is 19 at time $t=0$, inventory will be replenished when the cumulative number of units sold is 18. This occurs at time u, where $S(u)=e^{3u}-1=18\rightarrow u=.9815$.
The inventory being carried at time t for $t<.9815$ is $19-S(t)=19-e^{3t}+1=20-e^{3t}$.
The cost of carrying inventory at time t is $15(20-e^{3t})$. The total cost of carrying inventory from time $t=0$ to time $t=.9815$ is
$\int_0^{.9815}15(20-e^{3t})\,dt=15(20t-\frac{e^{3t}}{3})\Big|_{t=0}^{t=.9815}=204.4$. Answer: C

39. $M(t_1,t_2)=E[e^{t_1W+t_2Z}]=E[e^{t_1(X+Y)+t_2(Y-X)}]=E[e^{(t_1-t_2)X+(t_1+t_2)Y}]$
$=E[e^{(t_1-t_2)X}\cdot e^{(t_1+t_2)Y}]=E[e^{(t_1-t_2)X}]\cdot E[e^{(t_1+t_2)Y}]$
(this equality follows from the independence of X and Y)
$=M_X(t_1-t_2)\cdot M_Y(t_1+t_2)=e^{(t_1-t_2)^2/2}\cdot e^{(t_1+t_2)^2/2}=e^{t_1^2+t_2^2}$.
Answer: E

40. The speed is $\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = \sqrt{(2t-7)^2 + (\frac{t}{2}-6)^2} = \sqrt{\frac{17t^2}{4} - 34t + 85}$.
The speed will be minimized where $\frac{17t^2}{4} - 34t + 85$ is minimized for $t \geq 0$.
The critical point(s) occur where $\frac{17t}{2} - 34t = 0 \rightarrow t = 4$.
The speed at the critical point $t = 4$ is $\sqrt{17}$. The speed at the interval endpoint $t = 0$ is $\sqrt{85}$. As $t \rightarrow \infty$ the speed approaches ∞. The minimum speed occurs at $t = 4$.
Also, since the second derivative of $\frac{17t^2}{4} - 34t + 85$ is $\frac{17}{2} > 0$, it follows from the second derivative test that $t = 4$ is a local minimum. Answer: B

Answer: B